Deformations and Twisted Cohomology

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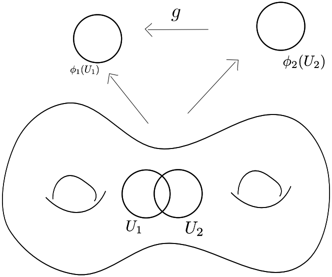
Overview

- Via developing, "Geometric Structures" on manifolds give rise to conjugacy classes of representations of fundamental groups into various lie groups
- Want to understand the space $\mathfrak{R}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)/G$, where Γ is finitely generated, G is a Lie group, and G acts by conjugation.
- Want to understand the space $\mathfrak{R}(\Gamma, G)$ locally near a class of representations $[\rho]$
- Want to understand the space $\mathfrak{R}(\Gamma, G)$ infinitesimally near a class of representations $[\rho]$.

(G, X)-structures

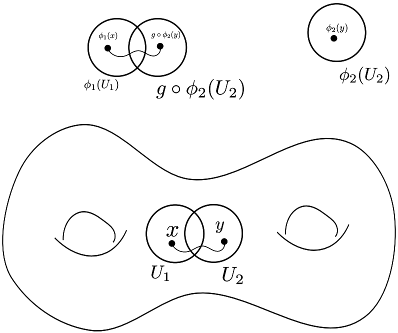
Let let G be a Lie group acting transitively on a simply connected manifold X, and let M be a manifold.

A (G, X)-structure on M is an atlas of charts $\{U_\phi\}$ such that $\phi_1 \circ \phi_2^{-1}$ agrees with an element of G on a neighborhood of each point in $U_{\phi_1} \cap U_{\phi_2}$.



Developing

Using *analytic continuation* we can globalize the data of a (G, X) structure and obtain a map $D : \tilde{M} \to X$



Holonomy

A developing map comes with a representation, $\rho : \pi_1(M) \to G$, with respect to which it is equivariant (i.e. $D(\gamma \cdot x) = \rho(\gamma) \cdot D(x)$).

Let $\gamma \in \pi_1(M)$ and τ_γ the corresponding deck transformation of $\tilde{M}$. $D \circ \tau_\gamma$ is a new developing map. Since $\tilde{M}$ is simply connected we see that $D \circ \tau_\gamma$ differs from D by post composition by a unique element of G, which we call $\rho(\gamma)$.

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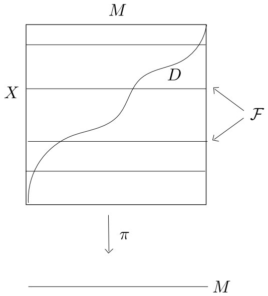
Using the above construction we see that the corresponding holonomies will differ by conjugation by an element of G. Thus to a (G, X) structure we associate a conjugacy class of representations.

The Associated Bundle

Given a (G, X) structure we can construct a bundle that encodes the developing map and holonomy. Let ρ be the holonomy, then we form the bundle $M \times_\rho X = (\tilde{M} \times X)/\pi_1(M)$ Here we are using the diagonal action $\gamma \cdot (m, x) = (\gamma \cdot m, \rho(\gamma) \cdot x)$.

This bundle is flat, has structure group G, and comes equipped with a foliation $\mathcal{F}$ coming from the quotient of $\tilde{M} \times \{x\}$ and a transverse section coming from D.



Reversing the Construction

Given a flat bundle $E \xrightarrow{\pi} M$ with a foliation $\mathcal{F}$ as above we can realize E as $M \times_\rho X$ for some representation ρ. Let

$[\gamma] \in \pi_1(M)$. Pick a basepoint $m_0 \in M$ and a path γ representing $[\gamma]$. Pick a point $e_0 \in E_0 = \pi^{-1}(x_0)$, then there is a lift $\tilde{\gamma}_{e_0}$ starting at e_0, lying over γ, and contained in a leaf of $\mathcal{F}$.

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The map $e_0 \to \tilde{\gamma}_{e_0}(1)$ gives an automorphism of E_0 (i.e. an element of G). Since E is flat this only depends on $[\gamma]$ and we get a map $\rho : \pi_1(M) \to G$.

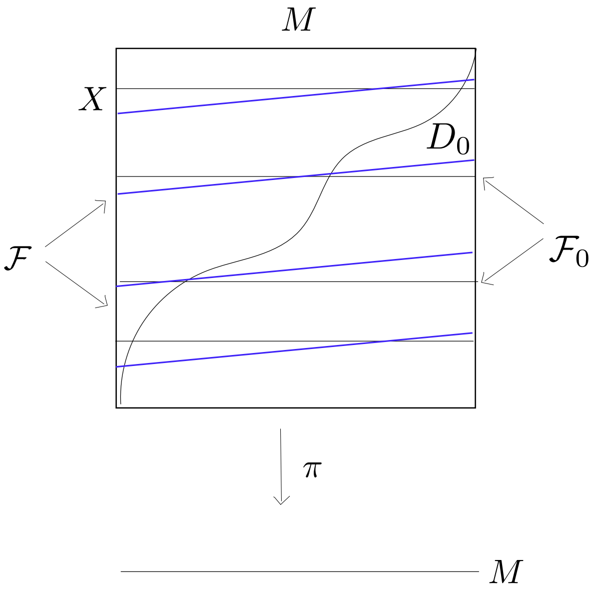
Realizing representations as (G, X)-structures

Using this bundle perspective we can show that if a representation ρ_0 is the holonomy of a (G, X)-structure on a compact manifold then nearby representations also come from (G, X) structures.

Proof (Sketch)

Let ρ be near ρ_0 then by covering homotopy property $M \times_{\rho_0} X \cong M \times_{\rho} X$ and so the foliations $\mathcal{F}_0$ and $\mathcal{F}$ are "close". As M is compact the section D_0 is also transverse to $\mathcal{F}$ and so we have a (G, X) structure.

Proof by Picture



Representation Varieties

If Γ is finitely generated and G is a “nice” group, then the set, $\mathcal{R}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)$ is an algebraic variety.

More concretely, a presentation for Γ gives rise to a polynomial function $f : \mathbb{R}^n \to \mathbb{R}^m$, and $\mathcal{R}(\Gamma, G)$ is $f^{-1}(0)$.

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If $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathbb{Z}/n\mathbb{Z}$, then $f : \mathbb{R}^4 \to \mathbb{R}^5$ is given by $f(A) = (A^n - I, \det A - 1)$, where we think of $A \in \mathbb{R}^4$.

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If 0 is a regular value of f then $f^{-1}(0)$ is a manifold and the tangent space to $p \in f^{-1}(0)$ is given by $\ker(f_* |_p)$

Even if 0 is not a regular value we can think of these kernels as a tangent spaces for $\mathcal{R}(\Gamma, G)$

Character Varieties

The way we attempt to realize $\mathfrak{X}(\Gamma, G)$ as a variety is by looking at the algebra of polynomials on $\mathcal{R}(\Gamma, G)$, which are invariant under the action of G.

These invariant polynomials are generated by traces of elements of Γ, and when G is "nice" this construction gives rise to a variety.

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However, this variety is not always the same as $\mathfrak{R}(\Gamma, G)$

Character Varieties

continued

We need to exclude representations whose image is like

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

because they cannot be distinguished from the trivial representation by looking at traces.

To get a variety we need to restrict to the set $\mathcal{R}'(\Gamma, G)$ of "nice" representations. In this case the quotient $\mathfrak{R}'(\Gamma, G) := \mathcal{R}'(\Gamma, G)/G$ is a variety.

Twisted Cohomology

Let G be a group and M a G-module. Define a cochain complex $C^n(G; M)$ to be the set of all functions from G^n to M with differential $d_n : C^n(G; M) \to C^{n+1}(G; M)$ by

$$d\phi(g_1, g_2, \ldots, g_{n+1}) = g_1 \cdot \phi(g_2, \ldots, g_{n+1}) +$$

$$\sum_{i=1}^{n} (-1)^i \phi(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \phi(g_1, \ldots, g_n)$$

Then $H^n(G; M) = Z^n(G; M)/B^n(G; M)$ is the associated cohomology group, where $Z^n(G; M) = \ker d_n$ and $B^n = \mathrm{Im}\, d_{n-1}$.

Low Dimensional Examples

$H^0(G; M)$

$C^0(G; M)$ is the set of constant functions. If $z \in C^0(G; M)$ then

$$d(z)(g) = g \cdot m_z - m_z.$$

Therefore,

$$H^0(G; M) = Z^0(G; M) = \{m \in M \mid g \cdot m = m \,\forall g \in G\}.$$

So $H^0(G; M)$ is the set of elements invariant under the action of G.

Low Dimensional Examples

$H^1(G; M)$

If $z \in Z^1(G; M)$ then

$$z(g_1 g_2) = z(g_1) + g_1 \cdot z(g_2)$$

These maps are sometimes called *crossed homomorphisms*.

We have already seen that $B^1(G; M)$ consists of maps where $z(g) = g \cdot m_z - m_z$ for some $m \in M$.

A Simple Example

Let $\mathbb{Z}$ act by conjugation (i.e. trivially) on $\mathbb{R}$, then $B^1(\mathbb{Z}, \mathbb{R}) = 0$ and if $z \in Z^1(\mathbb{Z}, \mathbb{R})$ then

$$z(mn) = z(m) + z(n),$$

and so $H^1(\mathbb{Z}, \mathbb{R}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{R}) = \mathbb{R}$ is the tangent space to $\mathrm{Hom}(\mathbb{Z}, \mathbb{R}) = \mathfrak{R}(\mathbb{Z}, \mathbb{R}) = \mathbb{R}$

In general, H^1 can be thought of as a "tangent space" to $\mathfrak{R}'(\Gamma, G)$.

H^1 as a Tangent Space

Let $\rho_0 : \Gamma \to G$ be a representation, let $\mathfrak{g}$ be the lie algebra of G, and let Γ act on $\mathfrak{g}$, by $\gamma \cdot x = \mathrm{Ad}\rho_0(\gamma) \cdot x$.

Denote the resulting cohomology groups $H^*(\Gamma, \mathfrak{g}_{\rho_0})$

Let ρ_t be a curve of representations passing through ρ_0.

For $\gamma \in \Gamma$ we can use a series expansion to write

$$\rho_t(\gamma) = (I + z_\gamma t + O(t^2))\rho_0(\gamma),$$

where $z_\gamma \in \mathfrak{g}$.

In this way we can think of z as an element of $C^1(\Gamma, \mathfrak{g}_{\rho_0})$

H^1 as a Tangent Space

continued

Repeatedly using this expansion again we see that

$$\rho_t(\gamma_1\gamma_2) = (I + z_{\gamma_1\gamma_2} t + O(t^2))\rho_0(\gamma_1\gamma_2) \text{ and}$$

H^1 as a Tangent Space

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$$\begin{aligned}\rho_t(\gamma_1)\rho_t(\gamma_2) &= (I + z_{\gamma_1}t + O(t^2))\rho_0(\gamma_1)(I + z_{\gamma_2}t + O(t^2))\rho_0(\gamma_2) \\ &= (I + (z_{\gamma_1} + \gamma_1 \cdot z_{\gamma_2})t + O(t^2))\rho_0(\gamma_1\gamma_2)\end{aligned}$$

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Therefore $z_{\gamma_1\gamma_2} = z_{\gamma_1} + \gamma_1 \cdot z_{\gamma_2}$, and so ρ_t gives rise to an element of $Z^1(\Gamma; \mathfrak{g}_{\rho_0})$

In this way $Z^1(\Gamma, \mathfrak{g}_{\rho_0})$ is the tangent space to $\mathcal{R}(\Gamma, G)$.

H^1 as a Tangent Space

continued

If $\rho_t(\gamma) = g_t^{-1}\rho_0 g_t$, where $g_t \in G$ and $g_0 = I$, then

$$\rho_t(\gamma) = (I - ct + O(t^2))\rho_0(\gamma)(I + ct + O(t^2))$$

So for deformations of this type, $z_\gamma = \gamma \cdot c - c$, and so $z \in B^1(\Gamma; \mathfrak{g}_{\rho_0})$

In this way trivial curves of deformations give rise to 1-coboundaries, and so $H^1(\Gamma, \mathfrak{g}_{\rho_0})$ is the tangent space to $\mathfrak{R}'(\Gamma, G)$ at ρ_0.

Another Simple Example

Lets compute the dimension of $H^1(\mathbb{Z}^2, \mathfrak{sl}_2(\mathbb{C})_{\rho_0})$ where

$$\rho_0(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \rho_0(b) = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \omega \neq 0$$

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Using "implicit differentiation" at $t = 0$ on the relation

$$\rho_t(a)\rho_t(b) = \rho_t(b)\rho_t(a)$$

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Using the exact sequence

$$0 \to Z^0(\mathbb{Z}^2; \mathfrak{sl}_2(\mathbb{C})_{\rho_0}) \to C^0(\mathbb{Z}^2; \mathfrak{sl}_2(\mathbb{C})_{\rho_0}) \to B^1(\mathbb{Z}^2; \mathfrak{sl}_2(\mathbb{C})_{\rho_0}) \to 0$$

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Therefore,

$$\dim H^1(\mathbb{Z}^2, \mathfrak{sl}_2(\mathbb{C})_{\rho_0}) = 6 - 2 - 2 = 2$$

Consequences

Theorem (Weil 64)

If ρ_0 is infinitesimally rigid (i.e. $H^1(\Gamma, \mathfrak{g}_{\rho_0}) = 0$), then ρ_0 is locally rigid (i.e. representations sufficiently close to ρ_0 are all conjugate)

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Example

$f(x) = x^2$ gives rise to a variety that is a single point, but whose tangent space is 1-dimensional.

Rigidity Results

There are various situations where rigidity results are known to hold.

Theorem (Weil)

If M is a closed, hyperbolic manifold of dimension $n \geq 3$ and ρ_0 is a discrete, faithful representation of $\Gamma = \pi_1(M)$ then $H^1(\Gamma, \mathfrak{so}(n,1)_{\rho_0}) = 0$

Similar results hold for cocompact lattices in most other semi-simple Lie groups.

However when Γ is no longer cocompact then interesting flexiblility phenomena can occur.

Rigidity and Flexiblity

The previous result tells us that ρ_0 cannot be deformed in $\mathrm{PSO(n, 1)}$.

However, we can embed $\mathrm{PSO(n, 1)}$ into other Lie groups (e.g. $\mathrm{PSO(n + 1, 1)}$, $\mathrm{PSU(n, 1)}$, or $\mathrm{PGL_{n+1}(\mathbb{R})}$), and ask if it is possible to deform ρ_0 inside of this larger Lie group.

Rigidity and Flexibility

Examples

Quasi-Fuchsian Deformations

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Projective Deformations

Cooper, Long, and Thistlethwaite examined deformations into $\mathrm{PGL}_4(\mathbb{R})$ by computing $H^1(\Gamma, \mathfrak{sl}_4{}_{\rho_0})$ for all closed, hyperbolic, two generator manifolds in the SnapPea census.

A majority of these two generator manifolds were rigid, however about 1.4 percent were infinitesimally deformable, and of those several have been rigorously shown to deform.

Rigidity and Flexibility

Non-compact Case

When M is a non-compact, finite volume, hyperbolic manifold of dimension 3 there are always non-trivial, hyperbolic deformations near ρ_0, but only one whose peripheral elements map to parabolics

In this case, we can still ask if ρ_0 is locally rigid relative ∂M (i.e. peripheral elements of $\pi_1(M)$ are sent to “parabolic” elements of $\mathrm{PGL}_4(\mathbb{R})$).

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For the two-bridge links with rational number $5/2$, $8/3$, $7/3$, *and* $9/5$ *are locally rigid relative* ∂M *at* ρ_0

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Question

Are all two-bridge knots and links rigid in this sense?